DOUBLE PANTS DECOMPOSITIONS REVISITED

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Abstract. Double pants decompositions were introduced in [FN] together with a flip-twist groupoid acting on these decompositions. It was shown that flip-twist groupoid acts transitively on a certain topological class of the decompositions, however, recently Randich discovered a serious mistake in the proof. In this note we present a new proof of the result, accessible without reading the initial paper.

MSC classes: 57M50
Key words: Pants decomposition, Heegaard splitting, Curve complex.

1. Introduction

Double pants decompositions are introduced in [FN] as a union of two pants decompositions of the same surface. These decompositions are subject to certain transformations (called "flips" and "handle twists" generating a groupoid called "flip-twist groupoid", see Section 2 for the definitions).

The main result of [FN] is stating that the flip-twist groupoid acts transitively on a certain set of double pants decompositions (called "admissible double pants decompositions"). In the case of closed surfaces, these admissible pants decompositions can be characterised as ones corresponding to Heegaard splittings of a 3-sphere. In other words, the following theorem was proved in [FN]:

Main Theorem. *Let S be an orientable surface of genus g with n holes, where $2g+n > 2$. Then the flip-twist groupoid acts transitively on the set of all admissible double pants decompositions of S.*

It was shown by Randich in his Master Thesis [R] that the original argument in [FN] contains a serious mistake. In this short note we present a new proof of the transitivity theorem, thus confirming that the main result of [FN] holds true.

As the new proof is short and technically easy, we try to keep this note independent of [FN]: Section 2 contains all definitions necessary to formulate and prove the main theorem (Section 3 is devoted to the mistake in the old proof and is not necessary to establish the result).

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2. Definitions

2.1. **Pants decompositions.** Let $S = S_{g,n}$ be an oriented surface of genus g with n holes. A closed curve $a \in S$ is called *essential* if no component of $S \setminus a$ is a disc, $a \in S$ is called *simple* if a has no self-intersections. In this paper, by a *curve* on S we will always mean a simple closed essential curve considered up to a homotopy of S. Given two curves we always assume that there are no "unnecessary intersections" (i.e. the homotopy classes of the curves contain no representatives intersecting in the smaller number of points). We denote by $|a \cap b|$ the number of intersections of the curves a and b.

Definition 2.1 (Pants decomposition)**.** A *pants decomposition* P of S is a collection of non-oriented mutually disjoint curves decomposing P into pairs of pants (i.e., into spheres with 3 holes).

There are two important types of transformations acting on pants decompositions:

Definition 2.2 (Flips)**.** Let $P = \{c_1, \dots, c_k\}$ be a pants decomposition, and suppose that $c_i \in P$ belongs to two different pairs of pants. A *flip* of P (in c_i) is a substitution of c_i by any curve such that $|c_i' \cap c_j| = 0$ for $j \neq i$ and $c_i' \cap c_i = 2$ (see Fig. 1.a).

Definition 2.3 ($\mathcal{S}$-moves)**.** Let $P = \{c_1, \dots, c_k\}$ be a pants decomposition and let $c_i \in P$ be a curve which belongs to a unique pair of pants. An *$\mathcal{S}$-move* in c_i is a substitution of c_i by any curve c_i' such that $|c_i' \cap c_j| = 0$ for $j \neq i$ and $c_i' \cap c_i = 1$ (see Fig. 1.b).

Notice that flips and $\mathcal{S}$-moves are not defined uniquely, for example repeatedly applying a Dehn twist along the curve c_i will result in infinitely many choices for the result of the flip (respectively, $\mathcal{S}$-move).

It is shown by Hatcher and Thurston [HT] that flips and $\mathcal{S}$-moves act transitively on all pants decompositions of a given surface.

2.2. **Double pants decompositions.**

Definition 2.4 (Double pants decomposition)**.** A *double pants decomposition* (P^a, P^b) is a set of two pants decompositions P^a and P^b of the same surface.

Clearly, flips act on double pants decompositions (we pick up a curve in P^a or P^b and perform the corresponding flip of an ordinary pants decomposition).

To model $\mathcal{S}$-moves, [FN] considers the transformations called *handle twists*. To define them, we will use a notion of a *handle curve*:

Definition 2.5 (Handle curve)**.** We will say that a curve c on a surface S is a *handle curve* if at least one of the connected components of the surface $S \setminus c$ is a torus with one hole (a "handle"). All other curves will be called *non-handle curves*.

Definition 2.6 (Handle twists)**.** Let (P^a, P^b) be a double pants decomposition and let $c \in P_a \cap P_b$ be a handle curve. Let H be the handle cut out by c, and let $a_1 \in P^a$ and $b_1 \in P^b$ be the curves contained in H. A *handle twist* of a_1 along b_1 is a Dehn twist along b_1 applied to a_1, see Fig. 1.c.

Definition 2.7 (FT-groupoid)**.** By a *flip-twist groupoid* (or FT-groupoid) we mean the groupoid acting on double pants decompositions and generated by all flips and handle twists.

Definition 2.8 (FT-equivalent double pants decompositions)**.** Two double pants decompositions are called *FT-equivalent* if there is a sequence of flips and handle twists transforming one of these decompositions to another.

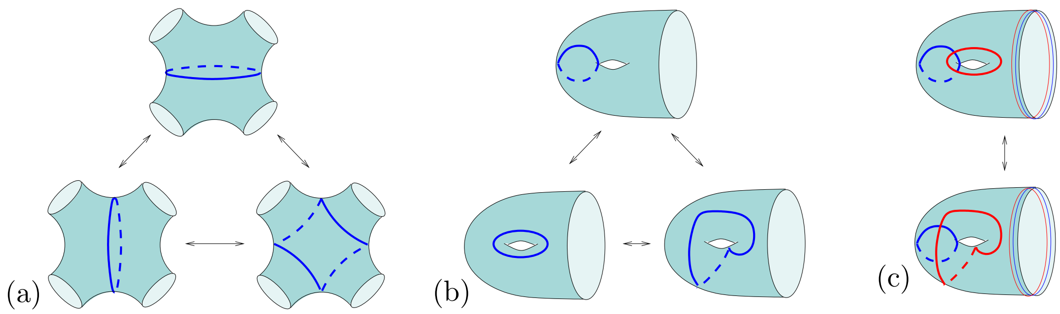

FIGURE 1. Examples of (a) flips; (b) $\mathcal{S}$-moves; (c) handle twist.

2.3. **Admissible double pants decompositions.**

Definition 2.9 (Standard double pants decompositions)**.** A double pants decomposition (P^a, P^b) of a genus g surface $S = S_{g,n}$ is *standard* if there is a set of handle curves $\{c_i\}$ in $P^a \cap P^b$ such that $S' = S \setminus \{c_i\}$ is a union of g handles $H_1, \dots, H_g$ and at most one sphere with holes, moreover, for $a_j, b_j \in H_j$, $a_j \in P^a$, $b_j \in P^b$ we require $|a_j \cap b_j| = 1$.

Definition 2.10 (Admissible double pants decompositions)**.** A double pants decomposition (P_1^a, P_1^b) is *admissible* if it may be obtained from a standard double pants decomposition by a sequence of flips.

Remark 2.11 (Admissible decompositions as Heegaard splittings of S^3). It is easy to show that in case of closed surfaces admissible double pants decompositions correspond to Heegaard splittings of 3-sphere, see [FN, Theorem 2.15] .

The main goal of [FN] and of the current note is to prove that any two admissible double pants decompositions are FT-equivalent.

3. THE ISSUE WITH THE OLD PROOF

The proof in [FN] was based on the notions of *zipper system* and *zipped flips* (see [FN, Definitions 1.3, 1.4, 1.7]). The idea was 1) to show that every admissible double pants decomposition is compatible with some zipper system; 2) to prove that all the decompositions compatible with the same zipper system are FT-equivalent; 3) to check that for all necessary changes of zipper systems one can use flips and handle twists.

In particular, the first of these steps was based on [FN, Lemma 1.12] which (wrongly) shows that every flip is a zipped flip.

In [R] Randich shows that the result of [FN, Lemma 1.12] is wrong: not every pants decomposition is compatible with a zipper system, and, consequently, not every flip is a zipped flip. To demonstrate this, Randich notices that if P is a pants decomposition compatible with a zipper system then the dual graph to P is planar, however, as Randich observes, this property is not always preserved by flips (see Fig. 2).

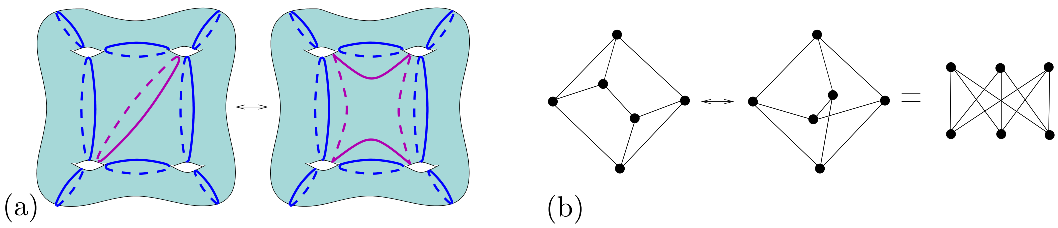

Figure 2. (a) An example of an unzipped flip; (b) The corresponding transformation of the dual graph results in a non-planar graph

Remark 3.1. In [FN, Section 3], the case of genus 2 was proved without usage of zipper systems, so, is not affected by the detected mistake. This gave the idea for construction of the new proof.

4. New proof of the Main Theorem

The proof is by induction on the genus of the surface (and on the number of holes for surfaces of the same genus).

Lemma 4.1. *Flips act transitively on double pants decompositions of $S_{0,n}$.*

Proof. As flips and $\mathcal{S}$-moves act transitively on (ordinary) pants decompositions [HT], and since no $\mathcal{S}$-moves are possible on the sphere, we conclude that flips act transitively on (ordinary) pants decompositions of $S_{0,n}$, as well as on double pants decompositions of $S_{0,n}$.

□

Lemma 4.1 settles the base of the induction, genus 0. From now on we consider a surface $S = S_{g,n}$ in assumption that $g > 0$ and that the main theorem holds for all surfaces $S' = S_{g',n'}$ satisfying $g' < g$ or $g' = g$, $n' < n$.

Lemma 4.2. *Let $q \subset S$ be a handle curve and suppose that $q \in P_1^a, P_1^b, P_2^a, P_2^b$, where (P_1^a, P_1^b) and (P_2^a, P_2^b) are two admissible double pants decompositions of S. Then (P_1^a, P_1^b) is FT-equivalent to (P_2^a, P_2^b).*

Proof. As q is a handle curve, it cuts S into two smaller surfaces. By the inductive assumption, all admissible double pants decompositions of $S \setminus q$ are FT-equivalent. Performing the same sequence of transformations on S, we see the FT-equivalence of all admissible double pants decompositions containing the same handle curve.

□

Lemma 4.3. *Let q_1 and q_2 be two distinct handle curves in S, $q_1 \cap q_2 = \emptyset$. Let (P_1^a, P_1^b) and (P_2^a, P_2^b) be admissible double pants decompositions of S such that $q_1 \in P_1^a \cap P_1^b$, $q_2 \in P_2^a \cap P_2^b$. Then (P_1^a, P_1^b) is FT-equivalent to (P_2^a, P_2^b).*

Proof. Cutting S along q_1 we obtain a handle and a surface S' of genus $g' < g$, hence by inductive assumption FT-groupoid acts transitively on the double pants decompositions of S'. In particular, (P_1^a, P_1^b) is FT-equivalent to a standard double pants decomposition (P_3^a, P_3^b) containing the curve q_2 (with $q_2 \in P_3^a \cap P_3^b$). In view of Lemma 4.2 (applied for $q = q_2$) this implies that (P_1^a, P_1^b) is FT-equivalent to (P_2^a, P_2^b).

□

Lemma 4.2 together with Lemma 4.3 motivate the following definition.

Definition 4.4 (FT-equivalent handle curves)**.** Let q_1 and q_2 be handle curves on S. We say that q_1 is *FT-equivalent* to q_2 if there exist admissible double pants decompositions (P_1^a, P_1^b) and (P_2^a, P_2^b) such that $q_1 \in P_1^a \cap P_1^b$, $q_2 \in P_2^a \cap P_2^b$, and (P_1^a, P_1^b) is FT-equivalent to (P_2^a, P_2^b).

In particular, Lemma 4.3 implies the following corollary.

Corollary 4.5. Any two disjoint handle curves in the same surface are FT-equivalent.

Lemma 4.6. *If S is a surface of positive genus, then for every non-handle curve $c \subset S$ there exists a handle curve $q \subset S$ such that $c \cap q = \emptyset$.*

Proof. If c is a separating curve (i.e. $S \setminus c$ is not connected), then at least one of the connected components of $S \setminus c$ is of positive genus, so, contains a handle curve q disjoint from c.

Now, suppose that c is not separating. Then there exists a curve a intersecting c at a unique point. Consider a neighbourhood of $c \cup a$: its boundary is a simple closed curve (denote it q, see Fig. 4.a). Moreover, it is easy to check that q is a handle curve, which is clearly disjoint from c.

□

Plan of proof of the theorem:

1. (Reduce to standard). As admissible double pants decompositions are the ones flip-equivalent to the standard ones, it is sufficient to show that any two standard double pants decompositions of the same surface are FT-equivalent.

2. (Reduce to one handle). In view of Lemma 4.2, any two standard double pants decompositions containing the same handle curve are FT-equivalent. So, to prove that any two standard double pants decompositions are FT-equivalent, it is sufficient to prove that any two handle curves c_{start} and c_{end} are FT-equivalent.

3. (On the curve complex, take a path from c_{start} to c_{end}). Consider the curve complex $\mathcal{C}(S)$ of S (i.e. the complex whose vertices correspond to homotopy classes of simple closed curves on S and whose simplices are spanned by vertices corresponding to disjoint sets of curves; in particular, the edges correspond to disjoint pairs of curves). In view of [HT] $\mathcal{C}(S)$ is connected, so, there exists a sequence σ of curves $\{c_{start} = c_0, c_1, c_2, \dots, c_m = c_{end}\}$ such that $c_i \cap c_{i+1} = \emptyset$ for $i = 1, \dots, m-1$.

4. (Decompose the path into handle-free subpathes). Decompose the sequence σ into finitely many subsequences $\sigma_1, \dots, \sigma_t$ such that the endpoints of each subsequence are handle curves and all other curves in σ are non-handle curves. It is sufficient to prove that the endpoints of one subsequence are FT-equivalent: Corollary 4.5 takes care of transferring from one subsequence to the adjacent one.

5. (Choose a disjoint handle for each curve in the subpath). Given a subsequence $\sigma_i = \{c_{i,1}, \dots, c_{i,m_i}\}$ (where $c_{i,1}$ and c_{i,m_i} are handle curves while all other curves in σ_i are not), for each non-handle curve $c_{i,j}$ $(1 < j < m_i)$ consider a handle curve $q_{i,j}$ which does not intersect $c_{i,j}$ (it does exist in view of Lemma 4.6). We get a caterpillar as in Fig 3 (sitting inside the curve complex).

6. (Move from one leg of the caterpillar to the adjacent one). It is left to prove that the handle curve $q_{i,j}$ is FT-equivalent to the handle curve $q_{i,j+1}$ (for any $1 \le j < m_i$). This is done in Lemmas 4.8 and 4.9.

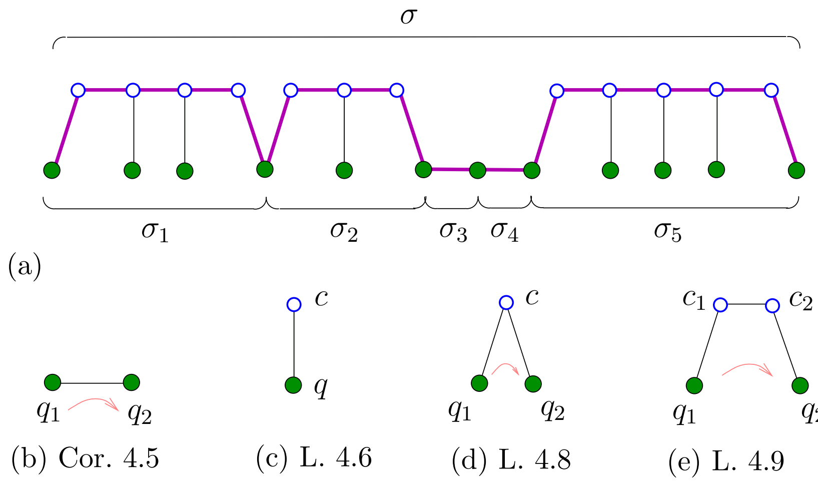

FIGURE 3. (a) Idea of proof: caterpillar; (b)-(e) List of lemmas. Filled/empty nodes denote handle/non-handle curves respectively.

Remark 4.7. The idea of a caterpillar-type proof is inspired by [FZ], where a caterpillar was used to prove Laurent phenomenon in cluster algebras.

Lemma 4.8. *Let $c \subset S$ be a non-handle curve, let $q_1, q_2 \subset S$ be two handle curves satisfying $c \cap q_1 = c \cap q_2 = \emptyset$. Then q_1 is FT-equivalent to q_2.*

Proof. Consider $S \setminus c$. If q_1 and q_2 belong to the same connected component of $S \setminus c$ then we use an inductive assumption (as all connected components are either of smaller genus or, in assumption of the same genus, have smaller number of holes). If q_1 and q_2 belong to different connected components, then $q_1 \cap q_2 = \emptyset$ and we can use Corollary 4.5.

□

Lemma 4.9. *Let $c_1, c_2 \subset S$ be two non-handle curves, $c_1 \cap c_2 = \emptyset$. Let $q_1, q_2 \subset S$ be handle curves such that $c_1 \cap q_1 = c_2 \cap q_2 = \emptyset$. Then q_1 is FT-equivalent to q_2.*

Proof. If $g > 2$ (where g is the genus of S), then at least one connected component of $S \setminus \{c_1, c_2\}$ is a surface of positive genus, so, there exists a handle curve $q \in S \setminus \{c_1, c_2\}$ which does not intersect $c_1 \cup c_2$. By Lemma 4.8, q_1 is FT-equivalent to q, and q is FT-equivalent to q_2, so the statement follows (see Fig. 4.b).

To prove the lemma for $g = 1, 2$, we will consider three cases: either both c_1 and c_2 are separating, or just one of them, or neither.

Case 1: both c_1 and c_2 are separating. Then $S \setminus \{c_1, c_2\}$ has a connected component of a positive genus, and, as above, there is a handle curve q in that component, such that $q \cap \{c_1 \cup c_2\} = \emptyset$. Thus, the statement follows again from Lemma 4.8 (see Fig. 4.b).

Case 2: c_1 is separating, c_2 is not separating. Consider $S' = S \setminus c_1$. Notice that the connected component of S' containing c_2 has a positive genus. So, by Lemma 4.6 there exists a handle curve $q \subset S'$ disjoint from c_2. Since q is also disjoint from c_1, we may apply Lemma 4.8 as in Fig. 4.b again.

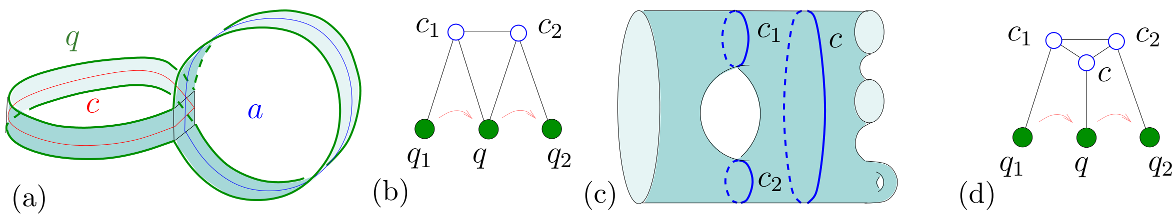

FIGURE 4. To the proof of Lemmas 4.6 and 4.9.

Case 3: neither c_1 nor c_2 are separating. We consider two possibilities: either $S' = S \setminus \{c_1, c_2\}$ is not a disjoint union $S_{0,3} \sqcup S_{0,3}$ of two pairs of pants or it is.

3.1. Suppose that $S' = S \setminus \{c_1, c_2\}$ is not a disjoint union of two pairs of pants (i.e. the surface is bigger than the one on Fig. 5.a). Then S' contains a separating curve c. If c is a handle curve, then we are in the settings of Fig. 4.b again (with $q = c$). If c is not a handle curve, then we can insert c into the sequence σ between c_1 and c_2 (as in Fig. 4d), use Lemma 4.6 to construct a handle curve q disjoint from c, and finally use Case 2 of the proof to show that q_1 is FT-equivalent to q and q is FT-equivalent to q_2.

3.2. Now, suppose that $S' = S \setminus \{c_1, c_2\}$ is a union of two disjoint pairs of pants, as in Fig. 5.a. Let q_1 and q_2 be the handle curves shown in Fig. 5.b and 5.c; notice that q_1 and q_2 are disjoint from c_1 and c_2 respectively. By Lemma 4.8, every handle curve in S' disjoint from c_1 is FT-equivalent to q_1, and every handle curve disjoint from c_2 is FT-equivalent to q_2. So, we are left to prove that q_1 is FT-equivalent to q_2, i.e. that there are double pants decompositions (P_1^a, P_1^b) and (P_2^a, P_2^b) such that $c_1 \in P_1^a \cap P_1^b$, $c_2 \in P_2^a \cap P_2^b$, and (P_1^a, P_1^b) is FT-equivalent to (P_2^a, P_2^b). An example of these double pants decompositions together with a sequence of FT-transformations is shown in Fig. 5.d.

$\square$

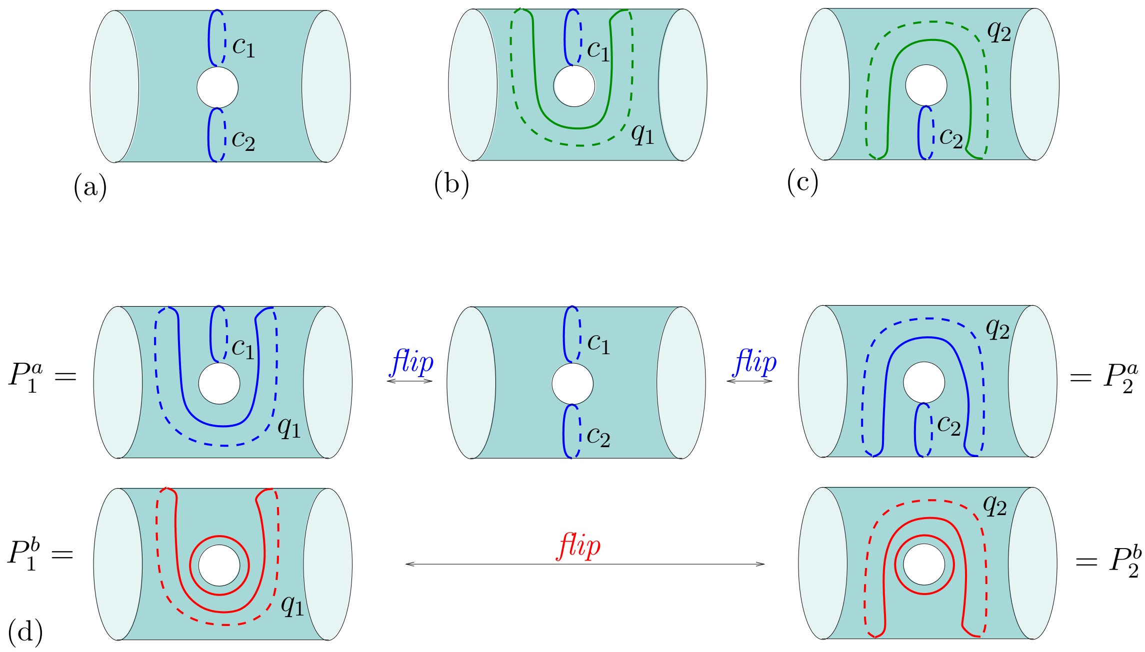

Figure 5. To the proof of Case 3.2.

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